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Quantum structure of the motion groups of the two-dimensional Cayley–Klein geometries

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Abstract. A simultaneous and global scheme of quantum deformation is defined for the set of algebras corresponding to the groups of motions of the two-dimensional Cayley–Klein geometries. Their central extensions are also considered under this unified pattern. In both cases some fundamental properties characterizing the classical CK geometries (as the existence of a set of commuting involutions, contractions and dualities relationships), remain in the quantum version.

1. Introduction

During the last few years, a wide range of literature has appeared about quantum groups (QG) (see [1–3] and references therein). The theory of quantum deformations of the universal enveloping algebras of semisimple Lie algebras has been studied rather extensively. However, for non-semisimple cases the results are partial. The most successful technique in order to get quantum deformations of non-semisimple algebras has been shown to be a generalized contraction method developed by Celeghini *et al* [4]. These authors have obtained [5, 6] a one-dimensional Heisenberg QG and a two-dimensional Euclidean QG, $E(2)_q$, by contraction of $SU(2)_q$. In [7] a three-dimensional Euclidean QG, $E(3)_q$, as well as some of its representations are obtained from $SO(4)_q$. By contracting again $E(3)_q$, a $(2+1)$ Galilei QG, $G(2+1)_q$, can be obtained. In the opposite sense, $E(3)_q$ and a $(2+1)$ Poincaré QG, $P(2+1)_q$, are realized by an expansion of $G(2+1)_q$. Throughout all these works, the essential feature is that the deformation parameter has to be transformed under contraction. Furthermore, physical applications of these algebras have been found by the same authors [8–11]. All these physical integrable models are strongly related with the presence of deformed commutation relations within the quantum algebra structure. Lukierski *et al* have also studied $(3+1)$ Poincaré quantum algebras starting from the real forms $U_q(O(3,2))$ of $U_q(Sl(4,\mathbb{C}))$ and following again a contraction method [12–14]. (See also [15] for another method to get consistent Hopf structures corresponding to some of these groups).

On the other hand, the classical versions of most of these groups appear as groups of motions of Cayley–Klein geometries (CKG). Recently a group-theoretical approach has been given for the N-dimensional CKG [16–18]; we want to use this scheme in order to build a unified pattern for the q-deformation of the Lie algebras and of the Lie groups of motions of these CKG. Throughout this paper the leading idea is to develop a scheme which might be extended to higher dimensions. Of course, the starting point is the $N=2$ case (whose geometries will be hereafter referred as 2D-CKG) where the groups of motions are $SO(3)$, $SO(2,1)$, the $(1+1)$ Galilei and Poincaré groups and the Euclidean plane group $E(2)$.

The non-simple groups in this set can be got by means of Inönü–Wigner [19] contractions, starting from $SO(3)$ and/or $SO(2,1)$.

By using a different procedure Man'ko and Gromov [20, 21] have studied the contractions of the irreducible representations of the so-called CKq unitary and CKq orthogonal algebras, i.e. a quantum version of the families of algebras obtained from $su(2)$ and $so(3)$ by contraction and analytic prolongations. The result of an Inönü–Wigner (IW) contraction is formally given in this case by simply replacing some 'parameters' from real or complex imaginary values to double ones. The CK algebras are thus quantized, but the contracted cases do not correspond to the quantizations of algebras given by Celeghini *et al* due to the fact that the deformation parameter is left unchanged under contraction. Another feature of the Man'ko and Gromov approach is that the contracted CK q-algebras can have no deformed commutation relations, and are therefore less suitable as q-symmetries of physical systems in the spirit of the above mentioned works.

In our global scheme of the 2D-CKG the CK groups and their related CKG are determined, as we shall see in section 2, by two real parameters (κ_1, κ_2) that have—in the classical case—a precise geometrical meaning: they are, respectively, the curvatures of the spaces of points and lines of the geometry. Consequently, our approach to the simultaneous quantization of the 2D-CK groups is more geometrical than that of Man'ko and Gromov [22, 23]. We start from the classical (non-deformed) CK algebras and we endow these algebras with a deformed Hopf structure [24], getting a simultaneous quantization of all of them. This deformation has an essential property: the specific characteristic relationships among the 2D-CKG appearing in the classical case (involutions, contractions and dualities) are preserved in their quantum version by defining a generalization of this underlying 'geometrical' structure. In particular, our generalized scheme prevents the deformed commutation relation to disappear and contains as generalized contractions the ones defined in [5–7]. When applied to the specific cases of $E(2)_q$, $G(1+1)_q$ and $P(1+1)_q$, our method leads to the results of Celeghini *et al.* It is interesting to mention that in their quantization and contraction procedures, these authors assume the rather natural commutative diagram:

$$\begin{array}{ccc} G_q & \stackrel{\varepsilon\to 0}{\longrightarrow} & CG_q \\ {\scriptstyle q\to 1}\downarrow & & {\scriptstyle q\to 1}\downarrow \\ G & \stackrel{\varepsilon\to 0}{\longrightarrow} & CG\,. \end{array}$$

Thus, the process of classical limit ($q \to 1$) that goes from the quantum group to its classical version commutes with the contraction limit ($\varepsilon \to 0$) that carries a (quantum) group to its contracted (quantum) version. In our approach, the Hopf structure is defined *a priori* for all the algebras, and this commutative diagram arises *after* the introduction of the whole CK structure is completed. Furthermore, a 'regular' description of contractions is also obtained, in the sense that the contracted group can be reached both as the 'limit' $\varepsilon \to 0$ of the pertinent family, and also considered by itself when one (or more) of the constants κ_i takes on zero values.

There is another interesting result that arises from our perspective: the equivalence between the emergence of a kind of contraction that transforms the deformation parameter and the interpretation of the latter—essential in the physical models quoted before—as a fundamental scale. Both statements are recovered at the same time from the quantum CK structure. From a physical point of view, the fundamental involutions are linked with parity and time reversal transformations. Since their quantum generalization includes an action of these automorphisms on the deformation parameter, this immediately acquires a dimensional interpretation.

We also present a quantization scheme for the central extensions of the above CK algebras. It is well known that the extended groups are sometimes the physically relevant ones in quantum mechanics. Thus, it is the central extension of the Galilei group $G(3+1)$ which gives rise to the mass of the physical system having Galilean symmetry; the pertinent realizations are the representations up to a factor of the Galilei group, whose equivalence classes are in relation with the factor systems [25]. Overall, the second cohomology group of any of these 2D-CK algebras [26] is either equal to $\mathbb{R}^2$, $\mathbb{R}$ or $\{0\}$ (this for the simple algebras $so(3)$ and $so(2,1)$). In order to present a global study and to shed light on the behaviour of these extensions under contraction, it turns out to be fruitful to consider extensions of all the CK algebras either by $\mathbb{R}^2$ or $\mathbb{R}$, though for some cases these extensions are trivial (yet become non-trivial by contraction [27]). In the quantum case, a global Hopf structure for all the extended CK algebras is given by considering as primitive one of the two generators of the central extensions.

This paper is organized as follows. Section 2 presents an overview about the main features of the 2D-CKG, paying special attention to the properties that will be conserved in the quantum case: involutions, geometrical structure related by contractions and dualities. We also describe the central extensions of all the CK algebras and we give a generalization of the above mentioned structure to this case. Section 3 develops in a detailed manner the quantization for both the non-extended and the extended CK algebras. In the next section we study the properties of the 2D-CKG that remain after quantum deformation, in both the non-extended and the extended cases. Finally some remarks relevant to the paper.

2. The Cayley–Klein algebras

We start this section by presenting an overview about the 2D-CKG in order to make this paper self-contained (see also [16, 18]). A more detailed and thorough exposition of the classical case is in preparation [17]. The second part deals with the central extensions of the CK groups. We give a detailed and global exposition; see also [25–30] where aspects of some cases are discussed separately.

2.1. The two-dimensional Cayley–Klein geometries

By a 2D-CKG we understand a geometrical system with two 2D-manifolds of points $\mathcal{X}^{(0)}$ and lines $\mathcal{X}^{(1)}$, both of which are symmetric homogeneous spaces of a three-dimensional Lie group G, relatively to a system of two commuting involutions of the Lie algebra $\mathfrak{g}$ of G. More precisely, the requirements are:

(1) There exist *two commuting involutions* $S_{(0)}$ and $S_{(1)}$ of the Lie algebra of G, $\mathfrak{g}$, with *one-dimensional* subalgebras of invariant elements $\mathfrak{h}^{(0)}$ and $\mathfrak{h}^{(1)}$, respectively.

(2) The group G acts transitively and effectively on the symmetric homogeneous spaces $\mathcal{X}^{(0)} \equiv G/H^{(0)}$ and $\mathcal{X}^{(1)} \equiv G/H^{(1)}$, where $H^{(0)}$ and $H^{(1)}$ are the subgroups corresponding to the subalgebras $\mathfrak{h}^{(0)}$ and $\mathfrak{h}^{(1)}$.

These two conditions suffice to characterize completely the 2D-CKG from a group-theoretical point of view. If we call J_{12} (resp P_1) the invariant element of $S_{(0)}$ (resp $S_{(1)}$), a basis of $\mathfrak{g}$ can be obtained by adding the invariant element P_2 under $S_{(2)} = S_{(0)} \cdot S_{(1)}$. In this basis, the involutions are given by

$$S_{(0)} : (P_1, P_2, J_{12}) \longrightarrow (-P_1, -P_2, J_{12}) \qquad S_{(1)} : (P_1, P_2, J_{12}) \longrightarrow (P_1, -P_2, -J_{12}) . \tag{2.1}$$

Note that the identity and the involutions $(S_{(0)}, S_{(1)}, S_{(2)})$ determine a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ Abelian group.

A set of measures of 'separation' between points and/or lines can be defined in a natural way; they are preserved under the action of G. It is easy to show that the systems satisfying the two requirements are characterized by two *real* parameters, each of which can be rescaled to either $1, 0$ or -1, so there are *nine* essentially different systems.

In fact, the general commutation relations for P_1, P_2 and J_{12} will be

$$[J_{12}, P_1] = A \qquad [J_{12}, P_2] = B \qquad [P_1, P_2] = C \tag{2.2}$$

where A, B and C are linear combinations of the basic generators. Making use of the facts that $S_{(0)}$ and $S_{(1)}$ are involutions and that G acts effectively on $\mathcal{X}^{(0)}$ and $\mathcal{X}^{(1)}$, (2.2) becomes

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -\kappa_2 P_1 \qquad [P_1, P_2] = \kappa_1 J_{12} \tag{2.3}$$

where κ_1 and κ_2 are two real parameters. So, each CKG is determined by (κ_1, κ_2), and we will denote $G_{(\kappa_1,\kappa_2)}$ ($\mathfrak{g}_{(\kappa_1,\kappa_2)}$) the CK group (algebra) with commutation relations given by (2.3).

A generic expression for the second-order Casimir of $\mathfrak{g}_{(\kappa_1,\kappa_2)}$ is

$$C = P_2^2 + \kappa_2 P_1^2 + \kappa_1 J_{12}^2 . \tag{2.4}$$

Table 1. The nine two-dimensional CK geometries.

	Measure of distance		
Measure of angle	Elliptic $\kappa_1 = 1$	Parabolic $\kappa_1 = 0$	Hyperbolic $\kappa_1 = -1$
Elliptic $\kappa_2 = 1$	Elliptic $SO(3)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = J_{12}$ $C = P_2^2 + P_1^2 + J_{12}^2$	Euclidean $ISO(2)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = 0$ $C = P_2^2 + P_1^2$	Hyperbolic $SO(2,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = -J_{12}$ $C = P_2^2 + P_1^2 - J_{12}^2$
Parabolic $\kappa_2 = 0$	Co-Euclidean $ISO(2)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = J_{12}$ $C = P_2^2 + J_{12}^2$	Galilean $IISO(1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = 0$ $C = P_2^2$	Co-Minkowskian $ISO(1,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = -J_{12}$ $C = P_2^2 - J_{12}^2$
Hyperbolic $\kappa_2 = -1$	Co-hyperbolic $SO(2,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = J_{12}$ $C = P_2^2 - P_1^2 + J_{12}^2$	Minkowskian $ISO(1,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = 0$ $C = P_2^2 - P_1^2$	Doubly hyperbolic $SO(2,1)$ $[J_{12}, P_1] = P_2$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = -J_{12}$ $C = P_2^2 - P_1^2 - J_{12}^2$

The geometrical meaning of the above hypotheses is as follows: let O be the coset of $H^{(0)}$ in $G/H^{(0)} \equiv \mathcal{X}^{(0)}$; the subgroup $H^{(0)} = \langle J_{12} \rangle$ is the isotopy group for the action of G on $\mathcal{X}^{(0)}$, and J_{12} is the infinitesimal generator of 'rotations' around this point O. The involution $S_{(0)}$ appears as the symmetry (half turn) around O. Rotations around other points are obtained by conjugation. Likewise, let l be the coset of $H^{(1)}$ in $G/H^{(1)} \equiv \mathcal{X}^{(1)}$; the subgroup $H^{(1)} = \langle P_1 \rangle$ is the isotopy group for the action of G on $\mathcal{X}^{(1)}$. When one considers the action of G on $G/H^{(0)} \equiv \mathcal{X}^{(0)}$, the 'first-kind' line l will appear as the trace of the point O under the subgroup $H^{(1)}$, hence the name of P_1 as the generator of translations along the line l. The involution $S_{(1)}$ corresponds to the reflection in the line l. The incidence of O and l is embodied in the commutativity of the associated involutions. The remaining generator, P_2, whose involution $S_{(2)}$ commutes with $S_{(0)}$ and $S_{(1)}$ is similarly associated to a 'second-kind' line l' through O and orthogonal to l; the incidence $O \in l'$ and the orthogonality between l and l' are again embodied in the commutativity of the associated involutions.

Furthermore, the symmetric space $\mathcal{X}^{(0)}$ (resp $\mathcal{X}^{(1)}$) has a canonical connection, whose curvature turns out to be constant and equal to κ_1 (resp κ_2). Finally we note the following relevant aspects.

(1) The sheaf of points on a first-kind line is an elliptical/parabolical/hyperbolical one according to whether κ_1 is greater than/equal to/less than zero.

(2) Idem for the sheaf of points through a second-kind line depending on $\kappa_1\kappa_2$.

(3) Idem for the sheaf of lines through a point now in concordance with κ_2.

So the three values $\kappa_1, \kappa_1\kappa_2$ and κ_2 are somewhat linked to the generators P_1, P_2 and J_{12} (see also expressions (2.8) and (2.9)).

Table 1 displays the nine 2D-CKG with κ_1 and κ_2 reduced to the 'standard' values $(1, 0, -1)$. It gives the motion group (as an abstract group), the Lie brackets and the second-order Casimir. It is worth remarking that the motion groups of 2D-CK systems fall into five isomorphic classes, as for some cases the choices of the involutions $S_{(0)}$, $S_{(1)}$ within an 'abstract' group can be made in two or more non-equivalent ways [31]. In other words, there are essentially different choices of the two homogeneous spaces $\mathcal{X}^{(0)}, \mathcal{X}^{(1)}$ on which the same 'abstract' group acts. Among them there are two simple groups: $SO(3)$ and $SO(2,1)$. They appear in the corners of table 1, which correspond to non-zero values of both curvatures. The other three can be obtained by contractions starting from the simple ones. They are: the 2D-Euclidean group, $E(2)$ (or $ISO(2)$), the 1+1 Galilei group, $G(1+1)$ (or $IISO(1)$), and the $1+1$ Poincaré group $P(1+1)$ (or $ISO(1,1)$).

Two kinds of IW contractions are singled out in a natural way in this scheme, which correspond to contractions 'around' a point and 'around' a line. The basic involutions $S_{(0)}$ and $S_{(1)}$ determine a direct sum of $\mathfrak{g}_{(\kappa_1,\kappa_2)}$ into the subspaces of invariant and anti-invariant elements; more precisely, each involution determines a $\mathbb{Z}_2$ grading of $\mathfrak{g}_{(\kappa_1,\kappa_2)}$ and altogether a $\mathbb{Z}_2^{\otimes 2}$. This structure induces in general new kinds of contractions, called graded ones [32–34], that can be specialized to the CK algebras [35]. At the same time they determine an IW contraction, obtained by keeping fixed the invariant elements, multiplying the anti-invariant ones by a parameter ε, and then taking the limit $\varepsilon \to 0$. So

$$\mathfrak{h}^{(0)}: \text{ local contraction}: \quad \mathbb{P}_1 = \varepsilon P_1,\ \mathbb{P}_2 = \varepsilon P_2,\ \mathbb{J}_{12} = J_{12} \qquad \varepsilon \to 0 \tag{2.5a}$$

$$\mathfrak{h}^{(1)}: \text{ axial contraction}: \quad \mathbb{P}_1 = P_1,\ \mathbb{P}_2 = \varepsilon P_2,\ \mathbb{J}_{12} = \varepsilon J_{12} \qquad \varepsilon \to 0 \tag{2.5b}$$

where $\mathbb{P}_1$, $\mathbb{P}_2$ and $\mathbb{J}_{12}$ stand for the new generators. It is a matter of simple calculation to check that the effect of the local contraction is to make $\kappa_1 = 0$ while keeping fixed the value of κ_2, and that an axial contraction keeps fixed κ_1 and makes $\kappa_2 = 0$. In the first case, the contracted geometry 'approximates' the old one in a neighbourhood of a point and in the second case, the contracted geometry is valid near a (first-kind) line in the original geometry, hence the name of these contractions.

In table 1, the effect of a local contraction is to move to the adjacent cell in the middle column, while an axial contraction moves to the middle row. For instance, Euclidean geometry is the local contraction of the Elliptic geometry and also of the hyperbolic geometry; the same happens for their motion groups (the Gel'fand triple $SO(3) \rightarrow E(2) \leftarrow SO(2,1)$). Similarly Galilean geometry appears as the axial contraction of both Euclidean and Minkowskian geometry ($E(2) \rightarrow G(1+1) \leftarrow P(1+1)$). Equivalent relations exist between the other rows and columns.

Another remarkable feature of the 2D-CKG is duality, which is a well known property in projective geometry; for dimension two it boils down to the complete equivalence of points and lines. However, when metrical properties enter the scheme, duality is not usually considered, and it is plainly clear that the properties of points and lines in, say, the Euclidean plane are rather different and preclude any equivalence. But we can still consider a transformation carrying points (or lines) of an original CKG into lines (or points) of another CKG—that we will call the dual of the former. At the Lie algebra level, consider a new choice of the three basic generators, $\mathbb{P}_1, \mathbb{P}_2, \mathbb{J}_{12}$, given in terms of the old ones by

$$\mathbb{P}_1 = -J_{12} \qquad \mathbb{P}_2 = -P_2 \qquad \mathbb{J}_{12} = -P_1 \,. \tag{2.6}$$

It is quite trivial to check that the new generators span a CK Lie algebra (2.3), and that the new values of the coefficients κ_1, κ_2 are equal to the old ones κ_2, κ_1 (in this order). From the algebraic point of view, this amounts to an interchange of the role of the two basic involutions $S_{(0)}, S_{(1)}$, and geometrically corresponds to taking as points in the dual geometry the (first-kind) lines in the old, and conversely. We can describe this interchange by introducing a Lie algebra isomorphism $\mathbb{D}_0 : \mathfrak{g}_{(\kappa_1,\kappa_2)} \rightarrow \mathfrak{g}_{(\kappa_2,\kappa_1)}$, defined by the action (2.6) on the generators and whose action on the coefficients is

$$\mathbb{D}_0(\kappa_1,\ \kappa_1\kappa_2,\ \kappa_2) = (\kappa_2,\ \kappa_1\kappa_2,\ \kappa_1) \,. \tag{2.7}$$

In table 1 the duality corresponds to the reflection in the main diagonal (the elliptic, Galilean and doubly hyperbolic geometries are autodual): dual geometries are in symmetric positions about this diagonal and have isomorphic groups of motion. This notion of duality can be generalized in the context of the 2D-CKG. A precise definition is given in [16, 17]; for our purposes, we will say that a duality is a Lie algebra isomorphism $\mathfrak{g}_{(\kappa_1,\kappa_2)} \rightarrow \mathfrak{g}_{(\kappa_1',\kappa_2')}$ which induces a permutation of the measure coefficients $(\kappa_1, \kappa_1\kappa_2, \kappa_2)$. In this case there are six permutations of these three coefficients and the set of dualities can be expressed in terms of three of them, the just-mentioned $\mathbb{D}_0$, $\mathbb{D}_1$ and $\mathbb{D}_2$. These two new dualities transform the Lie generators in the following way:

$$\begin{aligned} &\mathbb{D}_1(P_1, P_2, J_{12}) = (P_1, -\kappa_1 J_{12}, P_2) \qquad && \kappa_1 \neq 0 \\ &\mathbb{D}_2(P_1, P_2, J_{12}) = (P_2, -\kappa_2 P_1, J_{12}) \qquad && \kappa_2 \neq 0 \,. \end{aligned} \tag{2.8}$$

Note that, for $\kappa_1 = 0$ ($\kappa_2 = 0$), $\mathbb{D}_1$ ($\mathbb{D}_2$) are not automorphisms, so there is no restriction in assuming $\kappa_1^2 = 1$ ($\kappa_2^2 = 1$) whenever $\mathbb{D}_1$ ($\mathbb{D}_2$) are applied. In this case the action of both dualities on the coefficients is

$$\begin{aligned} &\mathbb{D}_1(\kappa_1,\ \kappa_1\kappa_2,\ \kappa_2) = (\kappa_1,\ \kappa_2,\ \kappa_1\kappa_2) \qquad && \kappa_1^2 = 1 \\ &\mathbb{D}_2(\kappa_1,\ \kappa_1\kappa_2,\ \kappa_2) = (\kappa_1\kappa_2,\ \kappa_1,\ \kappa_2) \qquad && \kappa_2^2 = 1 \,. \end{aligned} \tag{2.9}$$

It is possible to give an explicit realization D of the CK algebras by matrices of $gl(3,\mathbb{R})$. The expression for the infinitesimal generators is

$$D(P_1)=\begin{pmatrix}0&-\kappa_1&0\\1&0&0\\0&0&0\end{pmatrix}\qquad D(P_2)=\begin{pmatrix}0&0&-\kappa_1\kappa_2\\0&0&0\\1&0&0\end{pmatrix}$$
$$D(J_{12})=\begin{pmatrix}0&0&0\\0&0&-\kappa_2\\0&1&0\end{pmatrix}\tag{2.10}$$

where the associations $P_1 \leftrightarrow \kappa_1$, $P_2 \leftrightarrow \kappa_1\kappa_2$, $J_{12} \leftrightarrow \kappa_2$ appear again. This allows one to realize the 2D-CK groups as groups of linear transformations in $\mathbb{R}^3$. By exponentiation of the matrices (2.10) we obtain the one-parameter subgroups which are easily expressed in terms of a set of 'generalized' trigonometric functions with a second variable κ as a label. The (generalized) cosine $C_\kappa(x)$ and sine $S_\kappa(x)$ are defined by

$$C_\kappa(x)=\begin{cases}\cos\sqrt{\kappa}x & \text{if } \kappa>0\\ 1 & \text{if } \kappa=0\\ \cosh\sqrt{-\kappa}x & \text{if } \kappa<0\end{cases}\qquad S_\kappa(x)=\begin{cases}\frac{1}{\sqrt{\kappa}}\sin\sqrt{\kappa}x & \text{if } \kappa>0\\ x & \text{if } \kappa=0\\ \frac{1}{\sqrt{-\kappa}}\sinh\sqrt{-\kappa}x & \text{if } \kappa<0.\end{cases}\tag{2.11}$$

These generalized functions agree with the usual circular and hyperbolic functions for $\kappa=1$ and $\kappa=-1$, respectively. For $\kappa=0$ we have the 'parabolic' or Galilean trigonometric functions. The main value of this seemingly innocent extension of notation is that it allows a compact and clear way of writing many relations for classical and quantum CKG. We mention here only the most elementary properties:

$$\begin{aligned}&C_\kappa^2(x)+\kappa S_\kappa^2(x)=1\\&C_\kappa(x\pm y)=C_\kappa(x)C_\kappa(y)\mp\kappa S_\kappa(x)S_\kappa(y)\\&S_\kappa(x\pm y)=S_\kappa(x)C_\kappa(y)\pm S_\kappa(y)C_\kappa(x)\end{aligned}\tag{2.12}$$

and also

$$\begin{aligned}C_\kappa(x)&=\frac{e^{i\sqrt{\kappa}x}+e^{-i\sqrt{\kappa}x}}{2}\\S_\kappa(x)&=\frac{e^{i\sqrt{\kappa}x}-e^{-i\sqrt{\kappa}x}}{2i\sqrt{\kappa}}.\end{aligned}\tag{2.13}$$

On the other hand, as we have mentioned in the introduction, the physically relevant kinematical groups appearing in the Bacry–Lévy Leblond classification [36] are also CK groups. If we take P_1 and P_2 as the generators of the temporal and spatial translations, respectively, and J_{12} as the generator of the pure inertial transformations, the six 2D-CKG with $\kappa_2 \leqslant 0$ have a physical interpretation as one-dimensional kinematical geometries (this is kinematics in (1+1) spacetime). According to the pair (κ_1,κ_2) these are: oscillating Newton–Hooke $(1,0)$, Galilean $(0,0)$, expanding Newton–Hooke $(-1,0)$, oscillating de Sitter (or anti-de Sitter) $(1,-1)$, Minkowskian $(0,-1)$ and expanding de Sitter $(-1,-1)$.

The involutions $S_{(2)}$, $S_{(1)}$ and $S_{(0)}$ are now the discrete symmetries: time reversal T, parity P and PT. The local contraction ($\kappa_1 \to 0$) corresponds to the so-called spacetime contraction and the axial contraction ($\kappa_2 \to 0$) with the speed–space one. With other physical assignations of the generators of $\mathfrak{g}_{(\kappa_1,\kappa_2)}$ it is possible to obtain 10 out of the 11 kinematical groups. The static group is missing: it is not a CKG since its action on its space of lines is not effective.

2.2. Central extensions of the Cayley–Klein algebras

The non-trivial central extensions of a Lie algebra $\mathfrak{g}$ are determined by its second cohomology group $H^2(\mathfrak{g}, \mathbb{R})$. For the CK algebras under consideration, the second cohomology group is either $\{0\}$ (for the simple algebras $so(3)$ and $so(2,1)$), $\mathbb{R}$ (for $\mathcal{E}(2)$ and $\mathcal{P}(1,1)$) or $\mathbb{R}^2$ (for $\mathcal{G}(1,1)$).

From the point of view of physical applications, the extended groups often appear in relation with representation groups. Typically, the interesting representations of symmetry groups are projective ones, which can be linearized by using a representation group [37]. This group is an extension of the original group by the (dual of the) second cohomology group; in this case, this amounts to considering extensions of the algebra $\mathfrak{g}$ by either $\{0\}$, $\mathbb{R}$ or $\mathbb{R}^2$. Let us first consider an extension by $\mathbb{R}^2$ for all the cases, with the understanding that in particular situations the extensions might be trivial (an additional reason for doing this so is that often a contraction 'produces' a non-trivial extension from a trivial one; this underlies many aspects of the non-relativistic limit [27]). Therefore, we will denote by I_j ($j = 1, 2$) two new central generators, and take as the non-vanishing commutation relations for the extended CK algebra $\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$ the following ones:

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -\kappa_2 P_1 - \alpha_2 I_2 \qquad [P_1, P_2] = \kappa_1 J_{12} + \alpha_1 I_1 \tag{2.14}$$

with $\alpha_1, \alpha_2 \in \mathbb{R}$. Note that the extension with $\alpha_2 \neq 0$ is trivial if $\kappa_2 \neq 0$ (redefine $P_1 \longrightarrow P_1 + \alpha_2 I_2/\kappa_2$), and similarly, the extension with $\alpha_1 \neq 0$ is trivial if $\kappa_1 \neq 0$ ($J_{12} \longrightarrow J_{12} + \alpha_1 I_1/\kappa_1$). It is straightforward to prove that the second-order Casimir of $\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$ is

$$C = P_2^2 + \kappa_2 P_1^2 + \kappa_1 J_{12}^2 + 2\alpha_2 I_2 P_1 + 2\alpha_1 I_1 J_{12}. \tag{2.15}$$

The characteristic properties of the non-extended CKG described in the previous paragraph can be easily translated to this case. The involutive automorphisms of $\mathfrak{g}_{(\kappa_1,\kappa_2)}$ (equation (2.1)) are extended to involutive automorphisms of $\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$ defined by

$$\begin{aligned} \tilde{S}_{(0)} &: (P_1, P_2, J_{12}, I_2, I_1) \longrightarrow (-P_1, -P_2, J_{12}, I_1, -I_2) \\ \tilde{S}_{(1)} &: (P_1, P_2, J_{12}, I_2, I_1) \longrightarrow (P_1, -P_2, -J_{12}, -I_1, I_2). \end{aligned} \tag{2.16}$$

Obviously, the identity and the three automorphisms $\tilde{S}_{(0)}$, $\tilde{S}_{(1)}$, $\tilde{S}_{(2)} \equiv \tilde{S}_{(0)} \cdot \tilde{S}_{(1)}$ constitute again an Abelian group.

There are two basic contractions in $\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$. By making the following substitutions:

Local contraction: $\mathbb{P}_1 = \varepsilon P_1 \qquad \mathbb{P}_2 = \varepsilon P_2 \qquad \mathbb{J}_{12} = J_{12} \qquad \mathbb{I}_1 = \varepsilon^2 I_1$

$\mathbb{I}_2 = \varepsilon I_2$ (2.17*a*)

Axial contraction: $\mathbb{P}_1 = P_1 \qquad \mathbb{P}_2 = \varepsilon P_2 \qquad \mathbb{J}_{12} = \varepsilon J_{12} \qquad \mathbb{I}_1 = \varepsilon I_1$

$\mathbb{I}_2 = \varepsilon^2 I_2$ (2.17*b*) in the commutation relations (2.14) and taking the limit $\varepsilon \to 0$ we obtain a contracted extended CK algebra with the coefficient $\kappa_1 = 0$ (local case) or $\kappa_2 = 0$ (axial one).

It is worth remarking that after the contraction (2.17) some trivial extensions become non-trivial ones. When it is separately considered, only the Galilei case $G(1+1)$ has a true two-dimensional space of extensions; this group comes from contraction of groups which have a true one-dimensional space of extensions ($E(2)$ and $P(1+1)$) and these ones come in turn from contraction from the simple groups ($SO(3)$ and $SO(2,1)$).

The set of six dualities existing in the non-extended case can also be implemented in $\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$. We give here only the expression for the extended ordinary duality, $\tilde{\mathbb{D}}_0$, since it is the only one that will be used in the q-deformation of $\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$. Its action over the generators of $\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$, the κ_i coefficients and the extensions parameters α_i is

$$\begin{aligned} &\tilde{\mathbb{D}}_0(P_1, P_2, J_{12}, I_1, I_2) = (-J_{12}, -P_2, -P_1, -I_2, -I_1) \\ &\tilde{\mathbb{D}}_0(\kappa_1, \kappa_1\kappa_2, \kappa_2) = (\kappa_2, \kappa_1\kappa_2, \kappa_1) \qquad \tilde{\mathbb{D}}_0(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1)\,. \end{aligned} \tag{2.18}$$

It can be easily checked that $\tilde{\mathbb{D}}_0$ is an automorphism of $\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$.

3. The quantum Cayley–Klein algebras

We recall that an associative algebra A is said to be a Hopf algebra [24] if there exist two homomorphisms called coproduct ($\Delta : A \longrightarrow A \otimes A$) and co-unit ($\epsilon : A \longrightarrow \mathbb{C}$), as well as an antihomomorphism (the antipode $\gamma : A \longrightarrow A$) such that, $\forall a \in A$

$$(id \otimes \Delta)\Delta(a) = (\Delta \otimes id)\Delta(a) \tag{3.1}$$

$$(\mathrm{id} \otimes \epsilon)\Delta(a) = (\epsilon \otimes \mathrm{id})\Delta(a) = a \tag{3.2}$$

$$m((\mathrm{id} \otimes \gamma)\Delta(a)) = m((\gamma \otimes \mathrm{id})\Delta(a)) = \epsilon(a)1 \tag{3.3}$$

where m is the usual multiplication $m(a \otimes b) = ab$.

3.1. The quantum non-extended Cayley–Klein algebras

The universal enveloping algebra of the CK algebra $\mathfrak{g}_{(\kappa_1,\kappa_2)}$ is a (classical) Hopf algebra with coproduct, co-unit and antipode given by

$$\begin{aligned} &\Delta(X) = 1 \otimes X + X \otimes 1 \qquad \Delta(1) = 1 \otimes 1 \\ &\epsilon(X) = 0 \qquad \epsilon(1) = 1 \qquad \gamma(X) = -X \end{aligned} \tag{3.4}$$

where $X \in \{P_1, P_2, J_{12}\}$. An algebra element Y is said to be primitive if $\Delta(Y) = 1 \otimes Y + Y \otimes 1$.

To obtain a quantization [38, 39] of $U\mathfrak{g}_{(\kappa_1,\kappa_2)}$, we have to define a deformed Hopf structure on the completion of $U\mathfrak{g}_{(\kappa_1,\kappa_2)}$ (the universal enveloping algebra of $\mathfrak{g}_{(\kappa_1,\kappa_2)}$) denoted as $A = U\mathfrak{g}_{(\kappa_1,\kappa_2)}\hat{\otimes}\mathbb{C}[[z]]$ (where $\mathbb{C}[[z]]$ represents the associative algebra of formal power series on z and coefficients in $U\mathfrak{g}_{(\kappa_1,\kappa_2)}$). Such an algebra must be isomorphic (as Hopf algebra) to $U\mathfrak{g}_{(\kappa_1,\kappa_2)}$ when $z \to 0$.

Once the deformed coproduct has been given, ϵ and γ are derived from it by solving (3.2) and (3.3). On the other hand, the commutation relations associated with a given coproduct have to be compatible with it as a homomorphism. We introduce the following deformed comultiplication (with $q = e^z$):

$$\begin{aligned} \Delta(P_2) &= 1 \otimes P_2 + P_2 \otimes 1 \\ \Delta(P_1) &= e^{-\frac{1}{2}zP_2} \otimes P_1 + P_1 \otimes e^{\frac{1}{2}zP_2} \\ \Delta(J_{12}) &= e^{-\frac{1}{2}zP_2} \otimes J_{12} + J_{12} \otimes e^{\frac{1}{2}zP_2}. \end{aligned} \tag{3.5}$$

This definition is easily proved to be consistent with (3.1) and has the right 'classical' limit (all generators are primitive when $z \to 0$).

The deformed commutation relations are obtained by imposing Δ to be a homomorphism. This general condition is translated into three equations:

$$\begin{aligned} \Delta([J_{12}, P_2]) &= e^{-\frac{1}{2}zP_2} \otimes [J_{12}, P_2] + [J_{12}, P_2] \otimes e^{\frac{1}{2}zP_2} \\ \Delta([P_1, P_2]) &= e^{-\frac{1}{2}zP_2} \otimes [P_1, P_2] + [P_1, P_2] \otimes e^{\frac{1}{2}zP_2} \\ \Delta([J_{12}, P_1]) &= e^{-zP_2} \otimes [J_{12}, P_1] + [J_{12}, P_1] \otimes e^{zP_2} \end{aligned} \tag{3.6}$$

and the limit $z \to 0$ of their solutions have to be precisely (2.3). A simple Ansatz for a solution is to keep the 'classical' commutators for $[P_1, P_2]$ and $[J_{12}, P_2]$, and to try for the remaining commutator

$$[J_{12}, P_1] = \alpha(z)(e^{zP_2} - e^{-zP_2}) = g(z)\, S_{-z^2}(P_2). \tag{3.7}$$

This is indeed a solution of (3.6) for any function $g(z)$. Requiring $\lim_{z\to 0} g(z) S_{-z^2}(P_2) = P_2$, we must choose $g(z)$ such that $\lim_{z\to 0} g(z) = 1$. As we shall see in the next section, the simplest solution $g(z) = 1$ allows us to preserve formally $[J_{12}, P_1]$ under contraction processes. Therefore, we pose as deformed commutation relations associated with (3.5) the following ones:

$$[J_{12}, P_1] = S_{-z^2}(P_2) \qquad [J_{12}, P_2] = -\kappa_2 P_1 \qquad [P_1, P_2] = \kappa_1 J_{12}. \tag{3.8}$$

The standard q-numbers $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ (that in fact correspond to $g(z) = (2z)/(e^z - e^{-z})$) are now replaced by the generalized sine function (2.11). Note also that (3.8) guarantees the existence of a deformed set of commutation relations whatever the value of the coefficients κ_i. We shall emphasize this point below.

A straightforward calculation shows that, for $X \in \{P_1, P_2, J_{12}\}$ the co-unit ϵ and the antipode γ are

$$\epsilon(X) = 0 \qquad \gamma(X) = -e^{\frac{1}{2}zP_2}\, X\, e^{-\frac{1}{2}zP_2}. \tag{3.9}$$

The latter expression can be rewritten in matrix form by using (3.8) and expanding the antipode as an exponential of the adjoint action. Generalized trigonometric functions appear again as essential constituents of our structure:

$$\gamma \begin{pmatrix} P_1 \\ P_2 \\ J_{12} \end{pmatrix} = - \begin{pmatrix} C_{\kappa_1\kappa_2}(\frac{1}{2}z) & 0 & -\kappa_1 S_{\kappa_1\kappa_2}(\frac{1}{2}z) \\ 0 & 1 & 0 \\ \kappa_2 S_{\kappa_1\kappa_2}(\frac{1}{2}z) & 0 & C_{\kappa_1\kappa_2}(\frac{1}{2}z) \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ J_{12} \end{pmatrix}. \tag{3.10}$$

A second-order element belonging to the centre of the Hopf algebra A under (3.8) can be found. The computation is rather simple provided the essential properties (2.12)–(2.13) of the generalized trigonometric functions are known. The Casimir reads

$$C_z = 4C_{\kappa_1\kappa_2}\left(\tfrac{1}{2}z\right)\left[S_{-z^2}\left(\tfrac{1}{2}P_2\right)\right]^2 + (2/z)S_{\kappa_1\kappa_2}\left(\tfrac{1}{2}z\right)\{\kappa_2 P_1^2 + \kappa_1 J_{12}^2\}. \qquad (3.11)$$

The coincidence of the limit $z \to 0$ of C_z with (2.4) can be easily checked. The explicit form of the quantum algebra properties for the nine possible cases is displayed in table 2, where each q-algebra is described by (1) the name of the corresponding 'geometrical' algebra; (2) the pair (κ_1, κ_2); (3) the deformed commutation relations; (4) the matrix realization of the antipode acting on the column vector (P_1, P_2, J_{12}) and, finally, (5) the 'second-order' q-Casimir. The coproduct (3.5) and the co-unit (3.9) are the same for the nine cases.

Table 2. The nine three-dimensional quantum CK algebras.

q–elliptic $so(3)_q$; (1, 1) $[J_{12}, P_1] = S_{-z^2}(P_2)$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = J_{12}$ $-\begin{pmatrix} \cos(\frac{1}{2}z) & 0 & -\sin(\frac{1}{2}z) \\ 0 & 1 & 0 \\ \sin(\frac{1}{2}z) & 0 & \cos(\frac{1}{2}z) \end{pmatrix}$ $C_z = 4\cos(\frac{1}{2}z)S^2_{-z^2}(\frac{1}{2}P_2) + \frac{2}{z}\sin(\frac{1}{2}z)\left\{P_1^2 + J_{12}^2\right\}$	q–Euclidean $iso(2)_q$; (0, 1) $[J_{12}, P_1] = S_{-z^2}(P_2)$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = 0$ $-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{2}z & 0 & 1 \end{pmatrix}$ $C_z = 4S^2_{-z^2}(\frac{1}{2}P_2) + P_1^2$	q-hyperbolic $so(2,1)_q$; (−1, 1) $[J_{12}, P_1] = S_{-z^2}(P_2)$ $[J_{12}, P_2] = -P_1$ $[P_1, P_2] = -J_{12}$ $-\begin{pmatrix} \cosh(\frac{1}{2}z) & 0 & \sinh(\frac{1}{2}z) \\ 0 & 1 & 0 \\ \sinh(\frac{1}{2}z) & 0 & \cosh(\frac{1}{2}z) \end{pmatrix}$ $C_z = 4\cosh(\frac{1}{2}z)S^2_{-z^2}(\frac{1}{2}P_2) + \frac{2}{z}\sinh(\frac{1}{2}z)\left\{P_1^2 - J_{12}^2\right\}$
q–co-Euclidean $iso(2)_q$; (1, 0) $[J_{12}, P_1] = S_{-z^2}(P_2)$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = J_{12}$ $-\begin{pmatrix} 1 & 0 & -\frac{1}{2}z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $C_z = 4S^2_{-z^2}(\frac{1}{2}P_2) + J_{12}^2$	q–Galilean $iiso(1)_q$; (0, 0) $[J_{12}, P_1] = S_{-z^2}(P_2)$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = 0$ $-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $C_z = 4S^2_{-z^2}(\frac{1}{2}P_2)$	q–co-Minkowskian $iso(1,1)_q$; (−1, 0) $[J_{12}, P_1] = S_{-z^2}(P_2)$ $[J_{12}, P_2] = 0$ $[P_1, P_2] = -J_{12}$ $-\begin{pmatrix} 1 & 0 & \frac{1}{2}z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $C_z = 4S^2_{-z^2}(\frac{1}{2}P_2) - J_{12}^2$
q-co-hyperbolic $so(2,1)_q$; (1, −1) $[J_{12}, P_1] = S_{-z^2}(P_2)$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = J_{12}$ $-\begin{pmatrix} \cosh(\frac{1}{2}z) & 0 & -\sinh(\frac{1}{2}z) \\ 0 & 1 & 0 \\ -\sinh(\frac{1}{2}z) & 0 & \cosh(\frac{1}{2}z) \end{pmatrix}$ $C_z = 4\cosh(\frac{1}{2}z)S^2_{-z^2}(\frac{1}{2}P_2) - \frac{2}{z}\sinh(\frac{1}{2}z)\left\{P_1^2 - J_{12}^2\right\}$	q-Minkowskian $iso(1,1)_q$; (0, −1) $[J_{12}, P_1] = S_{-z^2}(P_2)$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = 0$ $-\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2}z & 0 & 1 \end{pmatrix}$ $C_z = 4S^2_{-z^2}(\frac{1}{2}P_2) - P_1^2$	q-doubly hyperbolic $so(2,1)_q$; (−1, −1) $[J_{12}, P_1] = S_{-z^2}(P_2)$ $[J_{12}, P_2] = P_1$ $[P_1, P_2] = -J_{12}$ $-\begin{pmatrix} \cos(\frac{1}{2}z) & 0 & \sin(\frac{1}{2}z) \\ 0 & 1 & 0 \\ -\sin(\frac{1}{2}z) & 0 & \cos(\frac{1}{2}z) \end{pmatrix}$ $C_z = 4\cos(\frac{1}{2}z)S^2_{-z^2}(\frac{1}{2}P_2) - \frac{2}{z}\sin(\frac{1}{2}z)\left\{P_1^2 + J_{12}^2\right\}$

If we take for P_2 the fundamental matrix representation (2.10) of the algebra, we obtain $S_{-z^2}(D(P_2)) = (S_{\kappa_1\kappa_2}(z)/z)D(P_2)$. This fact allows us to define the fundamental representation of the quantum CK algebra (obviously verifying (3.8)) by

$$
\begin{aligned}
D_q(P_2) &= D(P_2) \\
D_q(P_1) &= \sqrt{\frac{S_{\kappa_1\kappa_2}(z)}{z}}\, D(P_1) \\
D_q(J_{12}) &= \sqrt{\frac{S_{\kappa_1\kappa_2}(z)}{z}}\, D(J_{12})\,.
\end{aligned}
\tag{3.12}
$$

It is worth of remarking that this fundamental quantum representation coincides with the classical one (equation (2.10)) when either κ_1 or κ_2 (or both) are zero.

As far as physical applications are concerned, the co-Euclidean quantum algebra has already been shown to be relevant as a model to describe the symmetry of the harmonic chain [11]. The fusion of phonons has been derived from the quantum structure and the spacing of the chain is related in that model to the deformation parameter.

3.2. The quantum extended Cayley–Klein algebras

We shall discuss *two* possible approaches to deformations of the extended CK algebras.

Firstly, we consider a five-dimensional extended algebra $\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$, for all CK algebras and we deform the classical Hopf structure of $U\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$ making the ansatz that the new generators I_1, I_2, are chosen with the same coproduct as, respectively, J_{12} and P_1, in order to obtain deformed commutation relations with the correct classical limit. Both generators of the extensions are again required to be central. This approach leads to three subcases, depending on whether both I_1 and I_2, or only one of them (I_1 or I_2) is non-primitive.

The second main approach is to consider a four-dimensional extended algebra for all CK algebras, adding only *one* new generator (I_1 or I_2) according to any of the two following possibilities:

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -\kappa_2 P_1 \qquad [P_1, P_2] = \kappa_1 J_{12} + \alpha_1 I_1 \tag{3.13}$$

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -\kappa_2 P_1 - \alpha_2 I_2 \qquad [P_1, P_2] = \kappa_1 J_{12}\,. \tag{3.14}$$

(It is better not to drop the subscripts in I_1, I_2, α_1, α_2.) The first extension is again trivial if $\kappa_1 \neq 0$, while the second is trivial if $\kappa_2 \neq 0$. In both cases, it is possible to deform all extended CK algebras with a generator I_1 (I_2) which is required to be central and to have the same coproduct as J_{12} (P_1).

3.2.1. Hopf structures for five-dimensional extended algebras.

(a) I_1 and I_2 non-primitives. In principle, we may enlarge the non-extended algebra coproduct (3.5) for the extended algebra as follows:

$$
\begin{aligned}
\Delta(P_2) &= 1 \otimes P_2 + P_2 \otimes 1 \\
\Delta(P_1) &= \mathrm{e}^{-\frac{1}{2}zP_2} \otimes P_1 + P_1 \otimes \mathrm{e}^{\frac{1}{2}zP_2} \\
\Delta(J_{12}) &= \mathrm{e}^{-\frac{1}{2}zP_2} \otimes J_{12} + J_{12} \otimes \mathrm{e}^{\frac{1}{2}zP_2} \\
\Delta(I_1) &= \mathrm{e}^{-\frac{1}{2}zP_2} \otimes I_1 + I_1 \otimes \mathrm{e}^{\frac{1}{2}zP_2} \\
\Delta(I_2) &= \mathrm{e}^{-\frac{1}{2}zP_2} \otimes I_2 + I_2 \otimes \mathrm{e}^{\frac{1}{2}zP_2}\,.
\end{aligned}
\tag{3.15}
$$

We must check that (3.15) is an algebra homomorphism with classical limit given by (2.14), where I_1 and I_2 are central generators. This latter condition means that we have to impose

$$\Delta([P_2, I_i]) = [\Delta(P_2), \Delta(I_i)] = \Delta([J_{12}, I_i]) = 0 \qquad i = 1, 2. \tag{3.16}$$

However, (3.16) is already a strong constraint that it is only fulfilled if $\kappa_1 = \kappa_2 = 0$. This coproduct is compatible with the commutation relations

$$[J_{12}, P_2] = -\alpha_2 I_2 \qquad [P_1, P_2] = \alpha_1 I_1. \tag{3.17}$$

It remains to solve the homomorphism condition for the bracket

$$[J_{12}, P_1] = f(z, P_2). \tag{3.18}$$

It is easy to check that (3.18) requires $\alpha_1 = \alpha_2 = 0$, and in this case $f(z, P_2) = S_{-z^2}(P_2)$. Therefore, we recover the quantum non-extended Galilei algebra as the only possibility fulfilling (3.15).

(b1) I_1 primitive and I_2 non-primitive. Another natural candidate to coproduct with I_1 primitive and I_2 non-primitive is

$$\begin{aligned}
&\Delta(J_{12}) = 1 \otimes J_{12} + J_{12} \otimes 1 \qquad \Delta(I_1) = 1 \otimes I_1 + I_1 \otimes 1 \\
&\Delta(P_1) = e^{-\frac{1}{2}z\kappa_1 J_{12}} e^{-\frac{1}{2}z\alpha_1 I_1} \otimes P_1 + P_1 \otimes e^{\frac{1}{2}z\kappa_1 J_{12}} e^{\frac{1}{2}z\alpha_1 I_1} \\
&\Delta(P_2) = e^{-\frac{1}{2}z\kappa_1 J_{12}} e^{-\frac{1}{2}z\alpha_1 I_1} \otimes P_2 + P_2 \otimes e^{\frac{1}{2}z\kappa_1 J_{12}} e^{\frac{1}{2}z\alpha_1 I_1} \\
&\Delta(I_2) = e^{-\frac{1}{2}z\kappa_1 J_{12}} e^{-\frac{1}{2}z\alpha_1 I_1} \otimes I_2 + I_2 \otimes e^{\frac{1}{2}z\kappa_1 J_{12}} e^{\frac{1}{2}z\alpha_1 I_1}.
\end{aligned} \tag{3.19}$$

Now we have to require that (3.19) defines a homomorphism with central elements I_1, I_2, and with a set of deformed commutation relations consistent with (3.19), which are

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -\kappa_2 P_1 - \alpha_2 I_2 \qquad [P_1, P_2] = S_{-z^2}(\kappa_1 J_{12} + \alpha_1 I_1). \tag{3.20}$$

These requirements lead us clearly to a possible solution if we assume $\kappa_1 = 0$ (see appendix). This simplifies our coproduct which now reads

$$\begin{aligned}
&\Delta(J_{12}) = 1 \otimes J_{12} + J_{12} \otimes 1 \qquad \Delta(I_1) = 1 \otimes I_1 + I_1 \otimes 1 \\
&\Delta(P_1) = e^{-\frac{1}{2}z\alpha_1 I_1} \otimes P_1 + P_1 \otimes e^{\frac{1}{2}z\alpha_1 I_1} \\
&\Delta(P_2) = e^{-\frac{1}{2}z\alpha_1 I_1} \otimes P_2 + P_2 \otimes e^{\frac{1}{2}z\alpha_1 I_1} \\
&\Delta(I_2) = e^{-\frac{1}{2}z\alpha_1 I_1} \otimes I_2 + I_2 \otimes e^{\frac{1}{2}z\alpha_1 I_1}.
\end{aligned} \tag{3.21}$$

The commutation relations consistent with (3.21) are

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -\kappa_2 P_1 - \alpha_2 I_2 \qquad [P_1, P_2] = S_{-z^2}(\alpha_1 I_1). \tag{3.22}$$

The co-unit is defined as

$$\epsilon(X) = 0 \qquad X \in \{P_1, P_2, J_{12}, I_1, I_2\} \tag{3.23}$$

and the antipode is simplified due to the central nature of $e^{\pm\frac{1}{2}z\alpha_1 I_1}$:

$$\gamma(X) = -e^{\frac{1}{2}z\alpha_1 I_1} X\, e^{-\frac{1}{2}z\alpha_1 I_1} = -X \qquad X \in \{P_1, P_2, J_{12}, I_1, I_2\}\,. \tag{3.24}$$

The second-order invariant is given by

$$C_z = P_2^2 + \kappa_2 P_1^2 + 2\alpha_2 I_2 P_1 + 2S_{-z^2}(\alpha_1 I_1) J_{12}\,. \tag{3.25}$$

(b2) I_2 primitive and I_1 non-primitive. This case is quite similar to the last one. The proposed coproduct is

$$\begin{aligned}
&\Delta(P_1) = 1 \otimes P_1 + P_1 \otimes 1 \qquad \Delta(I_2) = 1 \otimes I_2 + I_2 \otimes 1 \\
&\Delta(P_2) = e^{-\frac{1}{2}z\kappa_2 P_1} e^{-\frac{1}{2}z\alpha_2 I_2} \otimes P_2 + P_2 \otimes e^{\frac{1}{2}z\kappa_2 P_1} e^{\frac{1}{2}z\alpha_2 I_2} \\
&\Delta(J_{12}) = e^{-\frac{1}{2}z\kappa_2 P_1} e^{-\frac{1}{2}z\alpha_2 I_2} \otimes J_{12} + J_{12} \otimes e^{\frac{1}{2}z\kappa_2 P_1} e^{\frac{1}{2}z\alpha_2 I_2} \\
&\Delta(I_1) = e^{-\frac{1}{2}z\kappa_2 P_1} e^{-\frac{1}{2}z\alpha_2 I_2} \otimes I_1 + I_1 \otimes e^{\frac{1}{2}z\kappa_2 P_1} e^{\frac{1}{2}z\alpha_2 I_2}
\end{aligned} \tag{3.26}$$

and the deformed commutation relations are

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -S_{-z^2}(\kappa_2 P_1 + \alpha_2 I_2) \qquad [P_1, P_2] = \kappa_1 J_{12} + \alpha_1 I_1\,. \tag{3.27}$$

Now the requirement of I_1, I_2 being central elements leads to a set of equations, which certainly have a solution for $\kappa_2 = 0$ (see also the appendix). This simplifies our coproduct again:

$$\begin{aligned}
&\Delta(P_1) = 1 \otimes P_1 + P_1 \otimes 1 \qquad \Delta(I_2) = 1 \otimes I_2 + I_2 \otimes 1 \\
&\Delta(P_2) = e^{-\frac{1}{2}z\alpha_2 I_2} \otimes P_2 + P_2 \otimes e^{\frac{1}{2}z\alpha_2 I_2} \\
&\Delta(J_{12}) = e^{-\frac{1}{2}z\alpha_2 I_2} \otimes J_{12} + J_{12} \otimes e^{\frac{1}{2}z\alpha_2 I_2} \\
&\Delta(I_1) = e^{-\frac{1}{2}z\alpha_2 I_2} \otimes I_1 + I_1 \otimes e^{\frac{1}{2}z\alpha_2 I_2}\,.
\end{aligned} \tag{3.28}$$

The commutation relations consistent with (3.28) are

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -S_{-z^2}(\alpha_2 I_2) \qquad [P_1, P_2] = \kappa_1 J_{12} + \alpha_1 I_1\,. \tag{3.29}$$

The co-unit is defined as in (3.23) and the antipode is

$$\gamma(X) = -e^{\frac{1}{2}z\alpha_2 I_2} X\, e^{-\frac{1}{2}z\alpha_2 I_2} = -X \qquad X \in \{P_1, P_2, J_{12}, I_1, I_2\}\,. \tag{3.30}$$

The second-order invariant is given by

$$C_z = P_2^2 + \kappa_1 J_{12}^2 + 2S_{-z^2}(\alpha_2 I_2) P_1 + 2\alpha_1 I_1 J_{12}\,. \tag{3.31}$$

3.2.2. Hopf structures for four-dimensional extended algebras.
(c1) I_1 primitive. We choose the coproduct

$$\begin{aligned} \Delta(J_{12}) &= 1 \otimes J_{12} + J_{12} \otimes 1 \qquad \Delta(I_1) = 1 \otimes I_1 + I_1 \otimes 1 \\ \Delta(P_2) &= e^{-\frac{1}{2}z\kappa_1 J_{12}} e^{-\frac{1}{2}z\alpha_1 I_1} \otimes P_2 + P_2 \otimes e^{\frac{1}{2}z\kappa_1 J_{12}} e^{\frac{1}{2}z\alpha_1 I_1} \\ \Delta(P_1) &= e^{-\frac{1}{2}z\kappa_1 J_{12}} e^{-\frac{1}{2}z\alpha_1 I_1} \otimes P_1 + P_1 \otimes e^{\frac{1}{2}z\kappa_1 J_{12}} e^{\frac{1}{2}z\alpha_1 I_1} . \end{aligned} \tag{3.32}$$

It is straightforward to prove that (3.32) defines a homomorphism with central element I_1 provided the following commutation relations are satisfied:

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -\kappa_2 P_1 \qquad [P_1, P_2] = S_{-z^2}(\kappa_1 J_{12} + \alpha_1 I_1) . \tag{3.33}$$

The co-unit is again identically zero and the antipode is

$$\gamma(X) = -e^{\frac{1}{2}z\kappa_1 J_{12}} e^{\frac{1}{2}z\alpha_1 I_1} X\, e^{-\frac{1}{2}z\kappa_1 J_{12}} e^{-\frac{1}{2}z\alpha_1 I_1} = -e^{\frac{1}{2}z\kappa_1 J_{12}} X\, e^{-\frac{1}{2}z\kappa_1 J_{12}} \tag{3.34}$$

with $X \in \{P_1, P_2, J_{12}, I_1\}$. The explicit form of the antipode in terms of generalized trigonometric functions is

$$\gamma \begin{pmatrix} P_1 \\ P_2 \\ J_{12} \\ I_1 \end{pmatrix} = - \begin{pmatrix} C_{\kappa_2}(\kappa_1 \frac{1}{2} z) & S_{\kappa_2}(\kappa_1 \frac{1}{2} z) & 0 & 0 \\ -\kappa_2 S_{\kappa_2}(\kappa_1 \frac{1}{2} z) & C_{\kappa_2}(\kappa_1 \frac{1}{2} z) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ J_{12} \\ I_1 \end{pmatrix} . \tag{3.35}$$

Recall that, at a classical level, the extensions by I_1 are trivial when $\kappa_1 \neq 0$. The remaining cases give rise now to quantizations of the extended Euclidean, Galilean and Minkowskian algebras. Physical applications of these deformed structures are worth studying, since it is known that the presence of I_1 corresponds in the Minkowskian and Galilei classical algebras to a uniform and constant force field built-in in the 'free' kinematics.

The deformed second-order Casimir is now

$$\begin{aligned} \mathcal{C}_z = 4C_{\kappa_2}\left(\frac{\kappa_1 z}{2}\right) \Big[\kappa_1 S^2_{-z^2\kappa_1^2}\left(\frac{1}{2} J_{12}\right) C_{-z^2}(\alpha_1 I_1) \\ + \frac{1}{2} S_{-z^2\kappa_1^2}(J_{12}) S_{-z^2}(\alpha_1 I_1) \Big] + \frac{2}{\kappa_1 z} S_{\kappa_2}\left(\kappa_1 \frac{z}{2}\right) [P_2^2 + \kappa_2 P_1^2] . \end{aligned} \tag{3.36}$$

(c2) I_2 primitive. For this subcase the coproduct is

$$\begin{aligned} \Delta(P_1) &= 1 \otimes P_1 + P_1 \otimes 1 \qquad \Delta(I_2) = 1 \otimes I_2 + I_2 \otimes 1 \\ \Delta(P_2) &= e^{-\frac{1}{2}z\kappa_2 P_1} e^{-\frac{1}{2}z\alpha_2 I_2} \otimes P_2 + P_2 \otimes e^{\frac{1}{2}z\kappa_2 P_1} e^{\frac{1}{2}z\alpha_2 I_2} \\ \Delta(J_{12}) &= e^{-\frac{1}{2}z\kappa_2 P_1} e^{-\frac{1}{2}z\alpha_2 I_2} \otimes J_{12} + J_{12} \otimes e^{\frac{1}{2}z\kappa_2 P_1} e^{\frac{1}{2}z\alpha_2 I_2} \end{aligned} \tag{3.37}$$

and the commutation relations, co-unit, antipode and Casimir are given by

$$[J_{12}, P_1] = P_2 \qquad [J_{12}, P_2] = -S_{-z^2}(\kappa_2 P_1 + \alpha_2 I_2) \qquad [P_1, P_2] = \kappa_1 J_{12} \tag{3.38}$$

$$\epsilon(X) = 0 \qquad \gamma(X) = -\mathrm{e}^{\frac{1}{2}z\kappa_2 P_1} X\, \mathrm{e}^{-\frac{1}{2}z\kappa_2 P_1} \qquad X = \{P_1, P_2, J_{12}, I_2\} \tag{3.39}$$

$$\gamma \begin{pmatrix} P_1 \\ P_2 \\ J_{12} \\ I_2 \end{pmatrix} = - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & C_{\kappa_1}(\kappa_2 \frac{1}{2} z) & \kappa_1 S_{\kappa_1}(\kappa_2 \frac{1}{2} z) & 0 \\ 0 & -S_{\kappa_1}(\kappa_2 \frac{1}{2} z) & C_{\kappa_1}(\kappa_2 \frac{1}{2} z) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \\ J_{12} \\ I_2 \end{pmatrix} \tag{3.40}$$

$$C_z = 4C_{\kappa_1}(\kappa_2 \frac{z}{2}) \left[\kappa_2 S^2_{-z^2\kappa_2^2}\left(\frac{1}{2} P_1\right) C_{-z^2}(\alpha_2 I_2) + \frac{1}{2} S_{-z^2\kappa_2^2}(P_1) S_{-z^2}(\alpha_2 I_2) \right] + \frac{2}{\kappa_2 z} S_{\kappa_1}\left(\kappa_2 \frac{z}{2}\right) \left[P_2^2 + \kappa_1 J_{12}^2\right]. \tag{3.41}$$

Remember that these extensions by I_2 are trivial for $\kappa_2 \neq 0$; in this case we recover the quantum non-extended relations with a primitive generator P_1. The three non-trivial extensions correspond to the quantum deformations of the 'absolute time' kinematics, which are the oscillating and expanding Newton–Hooke algebras and the Galilei algebra (in geometrical terms, the co-Euclidean, co-Minkowskian and Galilean algebras). For all of them, I_2 denotes—in the classical interpretation—the mass of the system. Finally, note that, in the classical case, this extended Newton–Hooke oscillating algebra is isomorphic to the oscillator algebra. Here this (classical) isomorphism again relates this quantum algebra with the deformed harmonic oscillator algebra defined in [6] and [40].

Another possibility, not studied here in detail, consists in taking two generators of the non-extended algebra as primitive, while the other one and the central extensions I_1 (or I_2) as non-primitive. It is easy to check that only two q-deformed four-dimensional extended algebras appear and both correspond to the Galilei case. One of these (extension by I_2) was originally obtained by the Firenze group [9, 10] and our basis is $\{B, T, P, M\} \to \{J_{12}, P_1, P_2, I_2\}$). In the classical case, this extended Galilei algebra describes the Galilean kinematical symmetry of a particle with mass (the mass operator is just I_2). The dynamical symmetry of magnons in the Heisenberg model has been described by using this quantum extended Galilei $(1 + 1)$ algebra. These authors have shown that the symmetry of the quantum group is completely equivalent to the Bethe ansatz for these systems and that the deformation parameter has again the physical meaning of chain spacing. The remaining q-deformation corresponds to the extension by I_1, with P_2, and J_{12} as primitive elements.

4. Fundamental structure of the 3D q-CK algebras

The underlying scheme of involutions, contractions and dualities characterizing the classical CK structure arises from the analysis of its geometrical meaning, as we have shown in section 2. Once we make a deformation of the CK Lie algebras, the basic geometrical notions such as homogeneous spaces, spaces of points and lines, curvatures, etc are no longer applicable; we are just dealing with a precise deformation of the CK algebras. The remarkable thing is that we can translate to the quantum case the whole non-deformed fundamental structure that characterizes the (classical) CKG. This fact has a rather simple but important consequence: the deformation parameter z also has to be transformed under the action of the quantum involutions, contractions and symmetries, and we have to study the properties of these transformations for the complete Hopf structure. The possibility that the deformation parameter has to be related with some dimensional quantity was firstly stated in the restricted context of contractions of quantum algebras [5]. This is actually the case for the physical systems for which these quantum algebras have been successfully introduced as symmetries [8–11]. The q-CK scheme supports this point of view and, at the same time, shows that the geometrical concepts that define the intrinsic structure of the classical CK algebras are also relevant in order to build the corresponding quantum objects.

In the following we will give the generalized notions of involutions, contractions and dualities compatible with the quantum deformations defined in the previous section. All of them agree with their 'classical' counterparts in the limit $q \to 1$. For the sake of clarity, we present separately the non-extended and extended cases.

4.1. Fundamental structure of the non-extended q-CK algebras

Involutive automorphisms. The involutions in $U_q\mathfrak{g}_{(\kappa_1,\kappa_2)}$ can be defined as follows:

$$\begin{aligned} &S^q_{(0)} : (P_1, P_2, J_{12}; z) \to (-P_1, -P_2, J_{12}; -z) \\ &S^q_{(1)} : (P_1, P_2, J_{12}; z) \to (P_1, -P_2 - J_{12}; -z)\,. \end{aligned} \tag{4.1}$$

The product $S^q_{(2)} = S^q_{(0)} S^q_{(1)}$ is also an involution which together with (4.1) constitutes an Abelian group $(\mathbb{Z}_2 \otimes \mathbb{Z}_2)$. These symmetries leave invariant the commutation relations (3.8) and the entire Hopf structure defined in 3.1.

Contractions. Two basic contractions also exist in $U_q\mathfrak{g}_{(\kappa_1,\kappa_2)}$. We define them in terms of the IW contractions by considering a transformation—dependent on a new parameter ε—of the generators and the deformation parameter. The transformed generators and the new deformation parameter are denoted by $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{J}_{12}$ and w, respectively. The explicit form of these two contractions is

$$q\text{-Local contraction:}\quad \mathbb{P}_1 = \varepsilon P_1 \qquad \mathbb{P}_2 = \varepsilon P_2 \qquad \mathbb{J}_{12} = J_{12} \qquad w = \frac{z}{\varepsilon} \quad \varepsilon \to 0 \tag{4.2}$$

$$q\text{-Axial contraction:}\quad \mathbb{P}_1 = P_1 \qquad \mathbb{P}_2 = \varepsilon P_2 \qquad \mathbb{J}_{12} = \varepsilon J_{12} \qquad w = \frac{z}{\varepsilon} \quad \varepsilon \to 0\,. \tag{4.3}$$

After writing the new commutation relations and the new Hopf structure, it can be easily shown that the limit $\varepsilon \to 0$ of (4.2) or (4.3) gives rise to a q-CK algebra with, respectively, $\kappa_1 = 0$ or $\kappa_2 = 0$. The composition of both transformations is also a contraction that makes both coefficients vanish. In order to show how this works, let us explicitly compute the q-local contraction.

We apply the contraction transformation (4.2) to the commutation relations:

$$\begin{aligned} &[\mathbb{J}_{12}, \mathbb{P}_1] = \varepsilon S_{-z^2}(P_2) = \varepsilon S_{-\varepsilon^2 w^2}(\mathbb{P}_2/\varepsilon) = S_{-w^2}(\mathbb{P}_2) \xrightarrow{\varepsilon\to 0} S_{-w^2}(\mathbb{P}_2) \\ &[\mathbb{J}_{12}, \mathbb{P}_2] = -\kappa_2 \varepsilon P_1 = -\kappa_2 \mathbb{P}_1 \xrightarrow{\varepsilon\to 0} -\kappa_2 \mathbb{P}_1 \\ &[\mathbb{P}_1, \mathbb{P}_2] = \varepsilon^2 \kappa_1 J_{12} = \varepsilon^2 \kappa_1 \mathbb{J}_{12} \xrightarrow{\varepsilon\to 0} 0\,. \end{aligned} \tag{4.4}$$

Note that the Lie bracket $[J_{12}, P_1]$ is formally preserved under contraction (due to the definition (2.11) of the generalized sine function). It is easy to check that any other choice for the arbitrary function $g(z)$ in (3.7) different from $g(z) = 1$ would not satisfy this property.

The coproduct and the co-unit are invariant under contraction (they are common for all the q-CK algebras). The antipode (3.10) is transformed in the following way:

$$\begin{aligned}
\gamma(\mathbb{P}_1) &= -\varepsilon C_{\kappa_1\kappa_2}(\tfrac{1}{2}z)P_1 + \kappa_1\varepsilon S_{\kappa_1\kappa_2}(\tfrac{1}{2}z)J_{12} = -C_{\kappa_1\kappa_2}(\varepsilon\tfrac{1}{2}w)\mathbb{P}_1 + \kappa_1\varepsilon S_{\kappa_1\kappa_2}(\varepsilon\tfrac{1}{2}w)\mathbb{J}_{12} \\
&= -C_{\kappa_1\kappa_2\varepsilon^2}(\tfrac{1}{2}w)\mathbb{P}_1 + \kappa_1\varepsilon^2 S_{\kappa_1\kappa_2\varepsilon^2}(\tfrac{1}{2}w)\mathbb{J}_{12} \\
\gamma(\mathbb{P}_2) &= -\varepsilon P_2 = -\mathbb{P}_2 \\
\gamma(\mathbb{J}_{12}) &= -\kappa_2 S_{\kappa_1\kappa_2}(\tfrac{1}{2}z)P_1 - C_{\kappa_1\kappa_2}(\tfrac{1}{2}z)J_{12} = -\frac{\kappa_2}{\varepsilon}S_{\kappa_1\kappa_2}(\varepsilon\tfrac{1}{2}w)\mathbb{P}_1 - C_{\kappa_1\kappa_2}(\varepsilon\tfrac{1}{2}w)\mathbb{J}_{12} \\
&= -\kappa_2 S_{\kappa_1\kappa_2\varepsilon^2}(\tfrac{1}{2}w)\mathbb{P}_1 - C_{\kappa_1\kappa_2\varepsilon^2}(\tfrac{1}{2}w)\mathbb{J}_{12}
\end{aligned} \tag{4.5}$$

and their $\varepsilon \to 0$ limit is just

$$\gamma\begin{pmatrix}\mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{J}_{12}\end{pmatrix} = -\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \kappa_2\frac{1}{2}w & 0 & 1\end{pmatrix}\begin{pmatrix}\mathbb{P}_1\\ \mathbb{P}_2\\ \mathbb{J}_{12}\end{pmatrix} \tag{4.6}$$

which coincides with making $\kappa_1 = 0$ in (3.10).

Dualities. The set of six dualities in $\mathfrak{g}_{(\kappa_1,\kappa_2)}$ can be generalized to $U_q\mathfrak{g}_{(\kappa_1,\kappa_2)}$ by defining suitable transformation properties of z and preserving the 'classical' effect over the generators and coefficients κ_i. The three basic dualities $\mathbb{D}_0^q$, $\mathbb{D}_1^q$ and $\mathbb{D}_2^q$ are displayed in table 3 which gives the transformed generators $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{J}_{12})$, deformation parameter (w) and measure coefficients (κ'_i). By means of these transformations, the 'dual' commutation relations, second-order Casimir and Hopf homomorphisms (writing for the antipode γ its associated matrix acting on the column vector $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{J}_{12})$) are obtained.

Two interesting remarks concerning the meaning of these quantum dualities can be made. First, the ordinary quantum duality $\mathbb{D}_0^q$ can be applied to the nine geometries and leaves the complete structure of $U_q\mathfrak{g}_{(\kappa_1,\kappa_2)}$ invariant. This means that the classical duality between points and lines in a pair of dual geometries is now translated into a duality of the deformed Hopf structures. Therefore, this duality is an automorphism in $U_q\mathfrak{g}_{(\kappa_1,\kappa_2)}$ and it establishes an isomorphism between quantum CK algebras preserving P_2 as primitive generator.

On the other hand, the remaining dualities can only be meaningfully applied to some geometries: to apply $\mathbb{D}_1^q$ ($\mathbb{D}_2^q$), κ_1 (κ_2) must be different from zero (a discussion of the geometrical meaning of these restrictions for the classical case is given in [16, 17]). Their effect on $U_q\mathfrak{g}_{(\kappa_1,\kappa_2)}$ consists of the change of the primitive generator. The classical dualities relate different geometries with the same CK Lie group. Their quantum analogues connect different possibilities to choose a primitive generator within the q-CK algebra. This is illuminated by considering any of the orthogonal q-CK algebras (i.e. $so(3)_q$ or $so(2,1)_q$), where all dualities can be applied. We may take any generator as primitive element and we shall find—in any case—a formally similar deformed Hopf structure and commutation relations. For the other algebras the restriction on the values of κ_i now prevents the deformed commutation relation from disappearing under the change of the primitive element. In this sense, these dualities define a kind of formal equivalence between deformed algebras.

Table 3. Deformed basic dualities acting on the three-dimensional q–CK algebras.

$\mathbb{D}_1^q;\ \kappa_1^2=1.$	$\mathbb{D}_0^q.$	$\mathbb{D}_2^q;\ \kappa_2^2=1.$
$\mathcal{P}_1=P_1$	$\mathcal{P}_1=-J_{12}$	$\mathcal{P}_1=P_2$
$\mathcal{P}_2=-\kappa_1 J_{12}$	$\mathcal{P}_2=-P_2$	$\mathcal{P}_2=-\kappa_2 P_1$
$\mathcal{J}_{12}=P_2$	$\mathcal{J}_{12}=-P_1$	$\mathcal{J}_{12}=J_{12}$
$w=z$	$w=-z$	$w=z$
$(\kappa_1',\kappa_1'\kappa_2',\kappa_2')=(\kappa_1,\kappa_2,\kappa_1\kappa_2)$	$(\kappa_1',\kappa_1'\kappa_2',\kappa_2')=(\kappa_2,\kappa_1\kappa_2,\kappa_1)$	$(\kappa_1',\kappa_1'\kappa_2',\kappa_2')=(\kappa_1\kappa_2,\kappa_1,\kappa_2)$
$[\mathcal{J}_{12},\mathcal{P}_1]=\mathcal{P}_2$	$[\mathcal{J}_{12},\mathcal{P}_1]=S_{-w^2}(\mathcal{P}_2)$	$[\mathcal{J}_{12},\mathcal{P}_1]=\mathcal{P}_2$
$[\mathcal{J}_{12},\mathcal{P}_2]=-\kappa_2'\mathcal{P}_1$	$[\mathcal{J}_{12},\mathcal{P}_2]=-\kappa_2'\mathcal{P}_1$	$[\mathcal{J}_{12},\mathcal{P}_2]=-\kappa_2' S_{-w^2}(\mathcal{P}_1)$
$[\mathcal{P}_1,\mathcal{P}_2]=\kappa_1' S_{-w^2}(\mathcal{J}_{12})$	$[\mathcal{P}_1,\mathcal{P}_2]=\kappa_1'\mathcal{J}_{12}$	$[\mathcal{P}_1,\mathcal{P}_2]=\kappa_1'\mathcal{J}_{12}$
$\Delta(\mathcal{P}_1)=\mathrm{e}^{-\frac{1}{2}w\mathcal{J}_{12}}\otimes\mathcal{P}_1+\mathcal{P}_1\otimes\mathrm{e}^{\frac{1}{2}w\mathcal{J}_{12}}$	$\Delta(\mathcal{P}_1)=\mathrm{e}^{-\frac{1}{2}w\mathcal{P}_2}\otimes\mathcal{P}_1+\mathcal{P}_1\otimes\mathrm{e}^{\frac{1}{2}w\mathcal{P}_2}$	$\Delta(\mathcal{P}_1)=1\otimes\mathcal{P}_1+\mathcal{P}_1\otimes 1$
$\Delta(\mathcal{P}_2)=\mathrm{e}^{-\frac{1}{2}w\mathcal{J}_{12}}\otimes\mathcal{P}_2+\mathcal{P}_2\otimes\mathrm{e}^{\frac{1}{2}w\mathcal{J}_{12}}$	$\Delta(\mathcal{P}_2)=1\otimes\mathcal{P}_2+\mathcal{P}_2\otimes 1$	$\Delta(\mathcal{P}_2)=\mathrm{e}^{-\frac{1}{2}w\mathcal{P}_1}\otimes\mathcal{P}_2+\mathcal{P}_2\otimes\mathrm{e}^{\frac{1}{2}w\mathcal{P}_1}$
$\Delta(\mathcal{J}_{12})=1\otimes\mathcal{J}_{12}+\mathcal{J}_{12}\otimes 1$	$\Delta(\mathcal{J}_{12})=\mathrm{e}^{-\frac{1}{2}w\mathcal{P}_2}\otimes\mathcal{J}_{12}+\mathcal{J}_{12}\otimes\mathrm{e}^{\frac{1}{2}w\mathcal{P}_2}$	$\Delta(\mathcal{J}_{12})=\mathrm{e}^{-\frac{1}{2}w\mathcal{P}_1}\otimes\mathcal{J}_{12}+\mathcal{J}_{12}\otimes\mathrm{e}^{\frac{1}{2}w\mathcal{P}_1}$
$\epsilon(\mathcal{P}_1)=\epsilon(\mathcal{P}_2)=\epsilon(\mathcal{J}_{12})=0$	$\epsilon(\mathcal{P}_1)=\epsilon(\mathcal{P}_2)=\epsilon(\mathcal{J}_{12})=0$	$\epsilon(\mathcal{P}_1)=\epsilon(\mathcal{P}_2)=\epsilon(\mathcal{J}_{12})=0$
$\Gamma=-\begin{pmatrix} C_{\kappa_2'}(\frac{1}{2}w) & S_{\kappa_2'}(\frac{1}{2}w) & 0 \\ -\kappa_2' S_{\kappa_2'}(\frac{1}{2}w) & C_{\kappa_2'}(\frac{1}{2}w) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\Gamma=-\begin{pmatrix} C_{\kappa_1'\kappa_2'}(\frac{1}{2}w) & 0 & -\kappa_1' S_{\kappa_1'\kappa_2'}(\frac{1}{2}w) \\ 0 & 1 & 0 \\ \kappa_2' S_{\kappa_1'\kappa_2'}(\frac{1}{2}w) & 0 & C_{\kappa_1'\kappa_2'}(\frac{1}{2}w) \end{pmatrix}$	$\Gamma=-\begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_1'}(\frac{1}{2}w) & \kappa_1' S_{\kappa_1'}(\frac{1}{2}w) \\ 0 & -S_{\kappa_1'}(\frac{1}{2}w) & C_{\kappa_1'}(\frac{1}{2}w) \end{pmatrix}$
$C_w=4C_{\kappa_2'}(\frac{1}{2}w)\left[S_{-w^2}(\frac{1}{2}\mathcal{J}_{12})\right]^2 + \frac{2}{w}S_{\kappa_2'}(\frac{1}{2}w)\left\{\kappa_1'\kappa_2'\mathcal{P}_1^2+\frac{1}{\kappa_1'}\mathcal{P}_2^2\right\}$	$C_w=4C_{\kappa_1'\kappa_2'}(\frac{1}{2}w)\left[S_{-w^2}(\frac{1}{2}\mathcal{P}_2)\right]^2 + \frac{2}{w}S_{\kappa_1'\kappa_2'}(\frac{1}{2}w)\left\{\kappa_2'\mathcal{P}_1^2+\kappa_1'\mathcal{J}_{12}^2\right\}$	$C_w=4C_{\kappa_1'}(\frac{1}{2}w)\left[S_{-w^2}(\frac{1}{2}\mathcal{P}_1)\right]^2 + \frac{2}{w}S_{\kappa_1'}(\frac{1}{2}w)\left\{\frac{1}{\kappa_2'}\mathcal{P}_2^2+\kappa_1'\kappa_2'\mathcal{J}_{12}^2\right\}$

4.2. Fundamental structure of the extended q-CK algebras

The pattern described above for the non-extended CK algebras can be reproduced for the extended ones. For the sake of brevity, in the sequel we summarize the main results about them.

Involutive automorphisms. For each of the five different quantizations defined in 3.2 there exists a commutative group of four symmetries (including the identity). These five sets of involutions have the same effect on the generators as in the classical case (see (2.16)). The only difference lies in the transformation properties of the deformation parameter z. We show the action of the basic involutions, $\tilde{S}^q_{(0)}$, $\tilde{S}^q_{(1)}$, on z:

Case	$\tilde{S}^q_{(0)}$	$\tilde{S}^q_{(1)}$	
b1/c1	$z \to z$	$z \to -z$	(4.7)
b2/c2	$z \to -z$	$z \to z$	

Contractions. The contraction procedures simply correspond to making $\kappa_i \to 0$; their complete effects on generators and on the deformation parameter z are displayed in table 4.

Table 4. Contractions of $U_q\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$.

Case	$\kappa_i \to 0$	Quantum algebra transformation					
$b1$	$\kappa_2 \to 0$	$\mathbb{J}_{12} = \varepsilon J_{12}$	$\mathbb{P}_1 = P_1$	$\mathbb{P}_2 = \varepsilon P_2$	$\mathbb{I}_2 = \varepsilon^2 I_2$	$\mathbb{I}_1 = \varepsilon I_1$	$w = z/\varepsilon$
$b2$	$\kappa_1 \to 0$	$\mathbb{J}_{12} = J_{12}$	$\mathbb{P}_1 = \varepsilon P_1$	$\mathbb{P}_2 = \varepsilon P_2$	$\mathbb{I}_2 = \varepsilon I_2$	$\mathbb{I}_1 = \varepsilon^2 I_1$	$w = z/\varepsilon$
$c1$	$\kappa_1 \to 0$	$\mathbb{J}_{12} = J_{12}$	$\mathbb{P}_1 = \varepsilon P_1$	$\mathbb{P}_2 = \varepsilon P_2$		$\mathbb{I}_1 = \varepsilon^2 I_1$	$w = z/\varepsilon^2$
	$\kappa_2 \to 0$	$\mathbb{J}_{12} = \varepsilon J_{12}$	$\mathbb{P}_1 = P_1$	$\mathbb{P}_2 = \varepsilon P_2$		$\mathbb{I}_1 = \varepsilon I_1$	$w = z/\varepsilon$
$c2$	$\kappa_1 \to 0$	$\mathbb{J}_{12} = J_{12}$	$\mathbb{P}_1 = \varepsilon P_1$	$\mathbb{P}_2 = \varepsilon P_2$	$\mathbb{I}_2 = \varepsilon I_2$		$w = z/\varepsilon$
	$\kappa_2 \to 0$	$\mathbb{J}_{12} = \varepsilon J_{12}$	$\mathbb{P}_1 = P_1$	$\mathbb{P}_2 = \varepsilon P_2$	$\mathbb{I}_2 = \varepsilon^2 I_2$		$w = z/\varepsilon^2$

Ordinary duality. The ordinary duality $\tilde{\mathbb{D}}^q_0$ acts in $U_q\tilde{\mathfrak{g}}_{(\kappa_1,\kappa_2)}$ as follows:

$$\tilde{\mathbb{D}}^q_0(P_1, P_2, J_{12}, I_1, I_2; z) = (-J_{12}, -P_2, -P_1, -I_2, -I_1; -z) \tag{4.8}$$

and it transforms the coefficients κ_i and α_i as in (2.18). On the other hand, $\tilde{\mathbb{D}}^q_0$ interchanges by pairs the following CK quantizations: (*b1*) with (*b2*) and (*c1*) with (*c2*).

5. Concluding remarks

The main result of this paper is the establishment of a systematic setting which allows a unified pattern of description of some q-deformations of CK algebras and their central extensions. Within this set of q-deformations, the deepest characteristics of the classical CKG survive and permit one to consider contractions, dualities, etc. We have discussed here at some length only the case $N = 2$, and even while some of the individual deformations of the non-extended groups have been found previously, we feel that the method is valuable, because of the structural scheme it affords. One rather appealing trait is the natural appearance of the deformation parameter as a quantity with physical dimensions. This is particularly clear in the 'kinematical' cases $\kappa_2 \leqslant 0$ where the basic involutions could be interpreted as the quantum versions of parity and time-reversal automorphisms. From (4.1)–(4.3), (4.7) and table 4, it follows that the deformation parameter is transformed under involutions and contractions in the same way as the primitive generator. In those cases where P_2 is primitive, it is therefore natural to assume that z has dimensions of length (recall that for $\kappa_2 \leqslant 0$, P_2 generates space translations). This is indeed consistent, as has been proved by the lattice models built by using the Poincaré and Galilei quantum algebras [9–11]. In fact, it is also possible to consider other two kinematical assignations of the geometrical generators with P_2 playing the role of time translation or inertial transformation, so that the possibility of finding different models where z might have different physical meanings is open [8].

Another remarkable but rather expected fact is that as one moves from the simple cases (both $\kappa_i \neq 0$) towards the $\kappa_i = 0$ algebras, more essentially different possibilities of deformation appear. This kind of 'degeneration' is indeed also present at the purely classical level.

In addition to these remarks, perhaps the main interest of this kind of approach lies in the possibility of its extension to higher values of N. In the classical CK groups this is possible and a nice scheme includes the simple groups as well as the r-quasisimple ones obtained by contraction, independently of the value and the parity of N. One could expect some differences between the even and odd cases for the quantum deformations of orthogonal groups, as they belong to different series of simple groups. The first odd case, $N = 3$, is attractive from a physical point of view, and a supply of different q-deformations of the $(2 + 1)$-dimensional kinematical groups would undoubtedly open the way to new model building. Furthermore, having a comprehensive idea of the first odd case would also clarify the way—or show the blocking difficulties—to an extension to higher dimensions. Work in this direction is in progress.

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Appendix

To show whether (3.19) is compatible with I_1 and I_2 being central elements we have to compute $\Delta([P_1, I_i])$, $\Delta([P_2, I_i])$, $\Delta([J_{12}, I_i])$ for $i = 1, 2$ and to impose these brackets to vanish under these central conditions. Since I_1 is a primitive element it is easy to check that all these requirements are fulfilled. However, the problems arise when we study the non-primitive extension I_2.

It is straightforward to see that $\Delta([J_{12}, I_2]) = 0$ if and only if $[J_{12}, I_2] = 0$. Hence, only the commutation relations between non-primitive elements remain to be studied. If we compute them we obtain

$$\begin{aligned}\Delta([P_i, I_2]) = {} & e^{-z\kappa_1 J_{12}}\, e^{-z\alpha_1 I_1} \otimes [P_i, I_2] + [P_i, I_2] \otimes e^{z\kappa_1 J_{12}}\, e^{z\alpha_1 I_1} \\ & + e^{-\frac{1}{2}z\kappa_1 J_{12}}\, e^{-\frac{1}{2}z\alpha_1 I_1} I_2 \otimes \left[P_i, e^{\frac{1}{2}z\kappa_1 J_{12}}\, e^{\frac{1}{2}z\alpha_1 I_1}\right] \\ & + \left[P_i, e^{-\frac{1}{2}z\kappa_1 J_{12}}\, e^{-\frac{1}{2}z\alpha_1 I_1}\right] \otimes e^{\frac{1}{2}z\kappa_1 J_{12}}\, e^{\frac{1}{2}z\alpha_1 I_1} I_2\end{aligned} \qquad \text{(A1)}$$

where $i = 1, 2$. To go on with the computation, we take into account the fact that a power-series form of the generalized trigonometric functions (2.11) is given by

$$S_\kappa(x) = \sum_{l=0}^{\infty}(-\kappa)^l \frac{x^{2l+1}}{(2l+1)!}$$
$$C_\kappa(x) = \sum_{l=0}^{\infty}(-\kappa)^l \frac{x^{2l}}{(2l)!} . \tag{A2}$$

It is possible to expand the brackets involving exponentials in (A1) and the resulting power series can be rewritten as generalized sine and cosine functions by means of (A2). The final expression reads

$$\begin{aligned}
\Delta[P_1, I_2] &= e^{-z\kappa_1 J_{12}} e^{-z\alpha_1 I_1} \otimes [P_1, I_2] + [P_1, I_2] \otimes e^{z\kappa_1 J_{12}} e^{z\alpha_1 I_1} \\
&\quad + e^{-\frac{1}{2}z\kappa_1 J_{12}} e^{-\frac{1}{2}z\alpha_1 I_1} I_2 \otimes \left\{(1 - C_{\kappa_2}(\kappa_1 \tfrac{1}{2}z))(\kappa_2 P_1 + \alpha_2 I_2)\right. \\
&\quad \left. - S_{\kappa_2}(\kappa_1 \tfrac{1}{2}z) P_2\right\} e^{\frac{1}{2}z\kappa_1 J_{12}} e^{\frac{1}{2}z\alpha_1 I_1} + \left\{(1 - C_{\kappa_2}(\kappa_1 \tfrac{1}{2}z))(\kappa_2 P_1 + \alpha_2 I_2)\right. \\
&\quad \left. + S_{\kappa_2}(\kappa_1 \tfrac{1}{2}z) P_2\right\} e^{-\frac{1}{2}z\kappa_1 J_{12}} e^{-\frac{1}{2}z\alpha_1 I_1} \otimes e^{\frac{1}{2}z\kappa_1 J_{12}} e^{\frac{1}{2}z\alpha_1 I_1} I_2 \\
\Delta[P_2, I_2] &= e^{-z\kappa_1 J_{12}} e^{-z\alpha_1 I_1} \otimes [P_2, I_2] + [P_2, I_2] \otimes e^{z\kappa_1 J_{12}} e^{z\alpha_1 I_1} \\
&\quad + e^{-\frac{1}{2}z\kappa_1 J_{12}} e^{-\frac{1}{2}z\alpha_1 I_1} I_2 \otimes \left\{(1 - C_{\kappa_2}(\kappa_1 \tfrac{1}{2}z)) P_2\right. \\
&\quad \left. + (\kappa_2 P_1 + \alpha_2 I_2) S_{\kappa_2}(\kappa_1 \tfrac{1}{2}z)\right\} e^{\frac{1}{2}z\kappa_1 J_{12}} e^{\frac{1}{2}z\alpha_1 I_1} + \left\{(1 - C_{\kappa_2}(\kappa_1 \tfrac{1}{2}z)) P_2\right. \\
&\quad \left. - (\kappa_2 P_1 + \alpha_2 I_2) S_{\kappa_2}(\kappa_1 \tfrac{1}{2}z)\right\} e^{-\frac{1}{2}z\kappa_1 J_{12}} e^{-\frac{1}{2}z\alpha_1 I_1} \otimes e^{\frac{1}{2}z\kappa_1 J_{12}} e^{\frac{1}{2}z\alpha_1 I_1} I_2 .
\end{aligned} \tag{A3}$$

Since $S_\kappa(0) = 0$, $C_\kappa(0) = 1$, $S_0(x) = x$ and $C_0(x) = 1$, the terms limited by keys must vanish to get I_2 as central element. Hence, κ_1 must be zero to preserve the homomorphism conditions (A3).

The proof is similar for the coproduct (3.26); now, the non-primitive extension is I_1, and we have to make $\kappa_2 = 0$ in order to get a Hopf algebra homomorphism. The reader may check that both results are related by the q-duality $\tilde{\mathbb{D}}_0^q$ defined in (4.8).

References

[1] Pressley A and Chari V 1990 *Nucl. Phys.* B *(Proc. Supl.)* **18** A 207
[2] Majid S 1991 *Int. J. Mod. Phys.* A **5** 1
[3] Zachos C 1992 *Contemp. Math.* **134** 351
[4] Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *Contractions of Quantum Groups (Lecture Notes in Mathematics 1510)* (Berlin: Springer) p 221
[5] Celeghini E, Giachetti R, Sorace E and Tarlini M 1990 *J. Math. Phys.* **31** 2548
[6] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 *J. Math. Phys.* **32** 1155
[7] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 *J. Math. Phys.* **32** 1159
[8] Bonechi F, Celeghini E, Giachetti R, Sorace E and Tarlini M *Phys. Lett.* **280B** 180
[9] Bonechi F, Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *Phys. Rev.* B **46** 5727
[10] Bonechi F, Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *J. Phys. A: Math. Gen.* **25** L939
[11] Bonechi F, Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 *Phys. Rev. Lett.* **68** 3178
[12] Lukierski J, Ruegg H, Nowicky A and Tolstoi V N 1991 *Phys. Lett.* **264B** 331

[13] Lukierski J, Ruegg H and Nowicky A 1991 *Phys. Lett.* **271B** 321
[14] Lukierski J, Ruegg H and Nowicky A 1992 *Phys. Lett.* **293B** 344
[15] Lyakhovsky V and Mudrov A 1992 *J. Phys. A: Math. Gen.* **25** L1139
[16] Herranz F J 1991 Geometrías de Cayley–Klein en *N* dimensiones y grupos cinemáticos *Tesina de Licenciatura* Universidad de Valladolid
[17] Herranz F J, del Olmo M A and Santander M, in preparation
[18] Santander M, Herranz F J and del Olmo M A 1993 Kinematics and homogeneous spaces for symmetrical contractions of orthogonal groups *Proc. XIX Int. Colloq. on Group Theoretical Methods in Physics CIEMAT (Madrid)* Vol. I p 455
[19] Inönü E and E P Wigner 1953 *Proc. Natl Acad. Sci. US* **39** 510
[20] Man'ko V I and Gromov N A 1992 *J. Math. Phys.* **33** 1374
[21] Gromov N A 1993 *J. Phys. A: Math. Gen.* **26** L5
[22] Man'ko V I and Gromov N A 1990 *J. Math. Phys.* **31** 1054
[23] Man'ko V I and Gromov N A 1990 *J. Math. Phys.* **31** 1060
[24] Abe E 1980 *Hopf Algebras (Cambridge Tracts in Mathematics 74)* (Cambridge: Cambridge University Press)
[25] Cariñena J F, del Olmo M A and Santander M 1985 *J. Math. Phys.* **26** 2096
[26] Gadella M, Martín M A, Nieto L M and del Olmo M A 1991 *J. Math. Phys.* **32** 1182
[27] Aldaya V and de Azcárraga A 1985 *Int. J. Theor. Phys.* **24** 141
[28] Lévy–Leblond J M 1971 *Group Theory and its Applications* vol 2 (New York: Academic)
[29] Lévy–Leblond J M 1969 *Commun. Math. Phys.* **12** 64
[30] Lévy–Leblond J M 1974 *Riv. Nuovo Cimento* **99** 265
[31] Cariñena J F, del Olmo M A and Santander M 1985 *J. Phys. A: Math. Gen.* **18** 1855
[32] de Montigny M and Patera J 1991 *J. Phys. A: Math. Gen.* **24** 525
[33] Patera J and Moody R V 1991 *J. Phys. A: Math. Gen.* **24** 2237
[34] de Montigny M, Patera J and Tolar J 1992 Graded contractions and kinematical groups of space-time *CRM preprint* Montréal
[35] Herranz F J, de Montigny M, del Olmo M A and Santander M 1993 Cayley–Klein geometries and graded contractions of $so(N+1)$ *UVA preprint* Valladolid
[36] Bacry H and M. Lévy-Leblond J 1968 *J. Math. Phys.* **9** 1605
[37] Cariñena J F and Santander M 1975 *J. Math. Phys.* **16** 1416; 1979 *J. Math. Phys.* **20** 1979
[38] Drinfeld V G 1986 Quantum groups *Proc. Int. Congr. of Mathematics, MRSI Berkeley* p 798
[39] Jimbo M 1985 *Lett. Math. Phys.* **10** 63; 1986 *Lett. Math. Phys.* **11** 247
[40] Gómez C and Sierra G 1993 *J. Math. Phys.* **34** 2119